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Poisson Point Processes, Cascades, and Random Coverings of R^n

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The generalized random energy model (GREM) is formulated in terms of hierarchies of Poisson point processes. This allows one to relate the high-temperature region with a random covering of R^n.

KEY WORDS: Generalized random energy model; Poisson point processes; hierarchies; random coverings.

1. INTRODUCTION

Random covering problems arise in several contexts. The main idea is to find the conditions for covering of a given set by smaller ones of fixed or variable size. An interesting problem is the study of the random variable representing the minimal number of sets necessary to cover. In the case of the noncomplete covering, another question concerns the properties of the set still uncovered.

First introduced by Dvoretzky,[(6)] the problem has been solved for the case of covering of the circle in refs. 9, 15, and 18, and in refs. 14 and 19 for the real line. In the case of a compact set, a very remarkable result is given in ref. 11 using potential theory. A recent result of Janson[(10)] generalizes the covering in two (and higher) dimensions; that work discusses the asymptotic distribution of the average number of covering sets.

In the physics literature, the covering of the real line by Poisson-distributed intervals is first encountered in ref. 1, where one-dimensional

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percolation models are studied using rigorous renormalization group methods. In particular, the absence of percolation is related to the covering of the line by given finite intervals called "dissociated" (see ref. 1 for details). On the other hand, an interpretation of the phase transition of the random energy model (REM) as the passage from a covering to a non-covering regime of the real line by Poisson-distributed intervals has been given recently by Koukiou.[14] In this case, the main idea is to interpret the "Boltzmann factors," in terms of which the partition function is written, as the lengths of the covering sets.

The purpose of this note is to pursue this work in the case of the generalized random energy model (GREM). Our motivation stems from the fact that the formulation of this model via Poisson point hierarchies allows one to discuss the covering in several dimensions. This is given in Section 2. For the convenience of the reader we recall in Section 3 some results of ref. 14 and we interpret the phase transitions of the GREM with n hierarchies as a random covering of R^n.

2. THE GREM AS POISSON CASCADES

For the study of spin-glass problems, two simplified models—the random energy model (REM) and the generalized random energy model (GREM)—have been introduced by Derrida and extensively studied in different contexts.[4,5] In the case of the REM, one has a system of independent identically distributed random variables E_i—the energy levels—and the partition function is written as the statistical sum over 2^N energy levels:

$$Z(\beta) = \sum_{i=1}^{2^N} \exp(-\beta E_i)$$

(β denotes the inverse temperature).

For the GREM, correlations between the energy levels are introduced in terms of hierarchies. More precisely, the 2^N configurations are grouped according to a hierarchy of n levels as follows. For any $n \in \mathbf{N}$ and $N \in \mathbf{N}$, let $\alpha_i \geqslant 1$ and $a_i \geqslant 0$, $i = 1,..., n$, be real, positive numbers such that $\sum_{i=1}^n a_i = 1$, $\sum_{i=1}^n \ln \alpha_i = \ln 2$. Consider the family of $\alpha_1^N + \alpha_1^N \alpha_2^N + \cdots + \alpha_1^N \cdots \alpha_n^N$ independent normalized Gaussian random variables $\varepsilon^j_{k_1,...,k_j}$, $j = 1,..., n$, $k_j = 1,..., \alpha_j^n$, defined on a probability space $(\Omega, \mathscr{F}, P)$. The energy levels are defined by

$$E_{k_1,...,k_n} = \sqrt{N} \sum_{j=1}^{n} \sqrt{a_j}\, \varepsilon^j_{k_i,...,k_j}$$

and the partition function is given by

$$Z(\beta)=\sum_{k_1=1}^{\alpha_1^N}\cdots\sum_{k_n=1}^{\alpha_n^N}\exp\left(\beta\sqrt{N}\sum_{j=1}^{n}\sqrt{a_j}\,\varepsilon^j_{k_1,\dots,k_j}\right)$$

These models have been a useful guide to understanding the thermodynamic behavior of the mean-field Sherrington–Kirkpatrick model.[20]

Recently, Ruelle reformulated these models in terms of Poisson distributions.[17] This approach is the starting point of ref. 14 and the present work. Some other rigorous results can be found in refs. 2, 7, and 8.

In the following, we reformulate the GREM in terms of *Poisson cascades*. Before defining these cascades, we recall some standard notations and definitions about Poisson point processes. For a general review see refs. 13 and 16.

Let X be a Borel space, $M_P(X)$ the family of point measures (i.e., sum of Dirac δ's) on $(X, \mathscr{B}(X))$, and $(\Omega, \mathscr{F}, P)$ a probability space.

Definition 1. A mapping $N: \Omega \to M_P(X)$ such that, for every $A \in \mathscr{B}(X)$, $N(\omega)(A)$ is $\mathscr{F}$-measurable, is called point process on X. The positive measure ν on $(X, \mathscr{B}(X))$, induced by N and given by $\nu(A) \equiv E(N(A)) = \int_\Omega N(\omega)(A)\, P(d\omega)$, is called the intensity of the process.

If $N(\omega)(A)$ is distributed according to a Poisson law with parameter $E(N(A))$, the process is called a *Poisson point process.*

A simple construction shows that for every σ-finite measure ν over $(X, \mathscr{B}(X))$, there always exists a concrete realization of a Poisson process, given by a sequence of random variables $(x_i)_{i\in\mathbf{N}}$ on X, such that:

(i) If A is a Borel subset of X, the number $N(A)$ of points $x_i \in A$ follows a Poisson law with expectation $\nu(A)$ [i.e., the measure ν is the intensity of the process; if $\nu(A)=\infty$, this is interpreted as $N(A)=\infty$ a.s.].

(ii) If the subsets $A_1,\dots, A_n$ are mutually disjoint, the random variables $N(A_1),\dots, N(A_n)$ are independent.

Let now $\Delta_n = (\mathbf{N}^*)^n$ be the set of sequences—of length n—of strictly positive integers. This set is in one-to-one correspondence with a rooted tree with n generations that is saturated (i.e., with infinite number of branches at each vertex). So, any given sequence $\mathbf{i} \in \Delta_n$ can be viewed as a particular branch of length n. For any $\mathbf{i} \in \Delta_n$, $\mathbf{i}\lceil_m$, with $m \leqslant n$, denotes the restriction of the sequence to its m first elements. Choose now an infinite sequence $\alpha = (\alpha_1,\dots, \alpha_n,\dots)$ of real nonnegative numbers and construct the family ν_{α_i} of measures on $\mathbf{R}\otimes\mathbf{R}^+$:

$$\nu_{\alpha_i} = \lambda \otimes \mu_{\alpha_i}$$

where λ denotes the Lebesgue measure on $\mathbf{R}$ and μ_{α_i} is the measure on $(\mathbf{R}^+, \mathscr{B}(\mathbf{R}^+))$ given by[17]

$$\mu_{\alpha_i}(l \in dy) = \alpha_i\, y^{-(1+\alpha_i)}\, dy$$

As was remarked in ref. 8, α_r corresponds to the ratio $\beta_{c,r}/\beta$, where $\beta_{c,r}$ denotes the inverse critical temperature of the rth hierarchy.

A *Poisson cascade* is defined recursively as follows:

Step 1. The first hierarchy is an infinity sequence of points $\{p_{i_1} \in \mathbf{R} \otimes \mathbf{R}^+,\ i_1 \in \mathbf{N}^*\}$ which are distributed according to ν_{α_1}. An equivalent way to consider this process is to define the random point measure

$$\mathcal{N}_{\alpha_1} = \sum_{i_1 \in \Delta_1} \delta_{p_{i_1}}$$

where δ_p is the Dirac measure concentrated on p. Here $\mathcal{N}_{\alpha_1}$ denotes the point measure of the Poisson point process of intensity ν_{α_1}.

Suppose now that this construction is carried up to the $n-1$ hierarchy.

Step n. For every point $p_{i_1,\dots,i_{n-1}} \in (\mathbf{R} \otimes \mathbf{R}^+)$, define the nth hierarchy as the infinite sequence of points $\{p_{i_1,\dots,i_n} \in \mathbf{R} \otimes \mathbf{R}^+,\ i_n \in N^*\}$ which are distributed according to the measure ν_{α_n}. The corresponding point measure is given by

$$\mathcal{N}_{\alpha_1,\dots,\alpha_n} = \sum_{\mathbf{i} \in \Delta_n} \delta_{p_{\mathbf{i}\restriction 1}} \delta_{p_{\mathbf{i}\restriction 2}} \cdots \delta_{p_{\mathbf{i}\restriction n}}$$

The point process defined by $\mathcal{N}_{\alpha_1,\dots,\alpha_n}$ is called an *n-Poisson cascade.*

We stress the fact that $\mathcal{N}_{\alpha_1,\dots,\alpha_n}$ is not a product of independent Poisson point processes [i.e., $\mathcal{N}_{\alpha_1,\dots,\alpha_n}$ is *not* the point measure of a process with intensity $(\lambda \otimes \mu_{\alpha_1}) \otimes (\lambda \otimes \mu_{\alpha_2}) \otimes \cdots (\lambda \otimes \mu_{\alpha_n})$].

Definition 2. The GREM with n hierarchies is an n-Poisson cascade.

In particular, the REM corresponds to a 1-Poisson cascade. Let us remark that in the above formulation we do not consider the states of the model in terms of probability measures. For this point the interested reader may consult ref. 17, where probability measures on the space of n hierarchies are constructed.

As was noticed in ref. 14, in order to have a geometric insight into the phase transition in the case of the REM, it was necessary to enhance it by an infinite number of real random parameters x_i. The intervals $]x_i, x_i + l_i[$ were identified with the random points $p_{\mathbf{i}}$ on $(\mathbf{R} \otimes \mathbf{R}^+)$, where l_i were the

Boltzmann factors $\exp(-\beta E_i)$. The same kind of enhancement will be necessary for the GREM. To each point $p_{i_1 \cdots i_n} = (x_{i_1 \cdots i_n}, l_{i_1 \cdots i_n})$ of $(\mathbf{R} \otimes \mathbf{R}^+)$, we naturally associate the open interval $]x_{i_1 \cdots i_n}, x_{i_1 \cdots i_n} + l_{i_1 \cdots i_n}[$, denoted for brievity $]p_{i_1 \cdots i_n}[$.

The open hypercubes

$$H^{(n)}_{i_1 \cdots i_n} =]p_{i_1}[\times]p_{i_1 i_2}[\times \cdots \times]p_{i_1 \cdots i_n}[$$

are the elementary tiles for which it will be examined whether they cover the space $\mathbf{R}^n$ a.s.

As we will see, the covering or noncovering depends on the parameters α_i, i.e., the temperature; we write explicitly the α_i dependence for the union

$$C(\alpha_1,..., \alpha_n) = \bigcup_{\mathbf{i} \in \Delta_n} H^{(n)}_{i_1 \cdots i_n}$$

The subset $C(\alpha_1,..., \alpha_n) \subset \mathbf{R}^n$ is called the *covered set*.

We also need the following family of auxiliary subsets $C^*(\alpha_r) \subset \mathbf{R}$, $r = 1,..., n$:

$$C^*(\alpha_r) = \bigcup_{i_r \in \mathbf{N}^*} H_{i_r}$$

In the following section we study the covering of $\mathbf{R}^n$ as a function of the parameters α_i.

3. THE HIGH-TEMPERATURE REGION AND THE RANDOM COVERING OF R^n

As introduction to this section, we recall some notations and results relative to the covering of the real line by Poisson intervals.

The covering condition can be found using different methods. Here, we use a recently developed approach related to the decomposition of positive measures on R into regular and singular parts. The main idea is to associate to the covering sets a positive martingale. This martingale can be considered as a sequence of random densities w.r.t. the Lebesgue measure, and one can investigate whether the weak limit coincides with a nontrivial measure.

The advantage of the method is that it allows are to obtain nontrivial results in a particularly simple manner. Similar methods can be applied to one-dimensional percolation to obtain a much simpler proof of several results of ref. 1. On the other hand, and more surprising, the study of the singularities of such a random measure has similarities with the phase transition problems of spin systems with random interactions.[3]

Let us now present the main lines of this construction (see ref. 12 for details).

For a locally compact space X, let $M^+(X)$ be the cone of positive Radon measures on X. Let $(\Omega, \mathscr{F}, P)$ be a probability space. For an increasing sequence of sub-σ-fields $(\mathscr{F}_n)_{n\in N} \subset \mathscr{F}$, consider the functions $(G_n(x, \omega))_{n\in \mathbf{N}}$ such that:

(i) For every $x \in X$, the sequence $(G_n(x, \cdot))_{n\in\mathbf{N}}$ is a positive (i.e., $\geqslant 0$), $\mathscr{F}_n$-adapted martingale.

(ii) For almost every $\omega \in \Omega$, the functions $G_n(\cdot, \omega)$ are Borel.

Let σ be an element of $M^+(X)$. For a Borel set $A \in X$, we consider the sequence of random measures

$$G_n\sigma(A) \equiv \int_A G_n(x, \omega)\, d\sigma(x)$$

Under the condition that the expectation $E(G_n(x, \omega)) \in L^1$, the above sequence converges a.s. to a random measure in the weak topology (Theorem 1 of ref. 12).

We distinguish two interesting cases. The first one is that this limit is zero a.s. In this case the measure σ is called *G_n-singular*. The second important case is when $E(\lim_{n\to\infty} G_n\sigma(\cdot)) = \sigma(\cdot)$; the measure σ is called *G_n-regular*. The decomposition is given by the following result.

Theorem 3.[12] Given a positive martingale G_n and a positive Radon measure σ on $(X, \mathscr{B}(X))$, there is a unique decomposition of G_n into a sum of two positive martingales

$$G_n = G_n^r + G_n^s$$

such that the measure σ is G_n^r-regular and G_n^s-singular.

We are now going to translate the above construction in terms of a covering problem.

Using the definitions of the previous section, we can ask whether $C^*(\alpha_r) = R$ or $C^*(\alpha_r) \neq \mathbf{R}$ almost surely.

Let F be a compact set, $F \subset \mathbf{R}$, and for an $\varepsilon > 0$ consider the regularized measure $\mu_{\alpha_r}^{\varepsilon}(\cdot) = \mu_{\alpha_r} \mathbb{1}_{]\varepsilon, 1]}(\cdot)$ ad the Poisson point process associated to $\nu_\varepsilon = \lambda \otimes \mu_{\alpha_r}^{\varepsilon}$. We define the functions

$$G_\varepsilon(x) = \frac{\mathbb{1}_{(x \notin C_\varepsilon^*(\alpha_r))}}{P(x \notin C_\varepsilon^*(\alpha_r))}, \qquad x \in \mathbf{R}$$

where $C_\varepsilon^*(\alpha_r)$ denotes the union of the covering intervals p_{i_r} with $l_{i_r} > \varepsilon$.

One can easily see that the above sequence defines a positive martingale of mean one, measurable w.r.t. the sequence of sub-σ-fields $\mathscr{F}_\varepsilon$ generated by $l_{i_r} > \varepsilon$. On the other hand, using the fact that Poisson point processes defined in disjoint domains are independent, the above martingale can be viewed as a product of independent random weights.

We have the following result.

Proposition 4. Almost surely:

(i) For $\alpha_r > 1$ we have

$$\lim_{\varepsilon \to 0} \sup_{x \in F} G_\varepsilon(x) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \to 0} G_\varepsilon \lambda(\cdot) = 0$$

In other words, the Lebesgue measure on $\mathbf{R}$ is G_ε-singular.

(ii) For $\alpha_r \leqslant 1$, $\lim_{\varepsilon \to 0} G_\varepsilon \lambda(\cdot) \neq 0$, the martingale $\int_0^1 G_\varepsilon(x)\, dx \in L^2(\Omega)$ and the Lebesgue measure is G_ε-regular.

The proof of this proposition can be found in ref. 14. Let us recall the main ideas.

(i) Remark first that a straightforward calculation allows one to write the martingale $G_\varepsilon(x)$ as

$$G_\varepsilon(x) = \mathbb{1}_{(x \notin C_\varepsilon^*(\alpha_r))} \exp \int_\varepsilon^\infty l \, d\mu_{\alpha_r}(l)$$

On the other hand, one can see that if the compact set $F \subset C_\varepsilon^*(\alpha_r)$, then the martingale $G_\varepsilon(x)$ indexed by the points of F should be degenerate, i.e.,

$$\lim_{\varepsilon \to 0} \sup_{x \in F} G_\varepsilon(x) = 0 \quad \text{a.s.}$$

As a consequence, $\lim_{\varepsilon \to 0} G_\varepsilon \lambda(F) = 0$ a.s. (λ denotes the Lebesgue measure). In fact, for $\alpha_r > 1$, the above martingale fails to be square integrable.

(ii) For $\alpha_r \leqslant 1$ the martingale $G_\varepsilon \lambda(F)$ converges in $L^2(\Omega)$ and the proposition follows using Doob's inequality for square-integrable martingales.

The following theorem is the main result of the present note. For clarity we assume that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$.

Theorem 5. We have:

(i) For $1 < \alpha_1$, $C(\alpha_1, ..., \alpha_n) = \mathbf{R}^n$ a.s.

(ii) If $\alpha_1 < \alpha_2 < \cdots < \alpha_j < 1 < \alpha_{j+1} < \cdots < \alpha_n$, for some $j \in \{1,..., n-1\}$, then any hyperplane of codimension j orthogonal to the first $1,..., j$-axes passing through any point of the covered set $C(\alpha_1,..., \alpha_j)$ is covered. All hyperplanes of codimension strictly smaller than j are uncovered; in particular, $\mathbf{R}^n$ is not covered.

(iii) For $\alpha_n \leqslant 1$ there is no hyperplane orthogonal to the axes covered.

Proof. (i) One can easily prove that $\forall j \in \{1,..., n-1\}$, the set $C^*(\alpha_r)$ covers the line iff the covered set $C(\alpha_1,..., \alpha_n) = \mathbf{R}^n$ a.s. Using this and the previous proposition, we have (i).

(ii) Applying the same reasoning to the hyperplanes orthogonal to the $1,..., j$-axes, (ii) follows.

(iii) For $\alpha_n \leqslant 1$, as the auxiliary family $C^*(\alpha_r)$ does not cover the line, we can easily conclude that there is no covered hyperplane. ∎

As the parameters α_r equal $\beta_{c,r}/\beta$, we can interpret geometrically the phase transitions of the GREM as follows. The case (i) corresponds to the high-temperature behavior of the model; the first j phase transitions are related to (ii), and the low-temperature region is related to (iii), where the system is completely frozen.

Let us remark that we cannot prove the above result if the distribution of the random covering intervals is not Poisson. An interesting direction is to apply the above setting to percolation problems. In this case we can use some recent results in order to have estimates on the lengths and the distribution of the number of covering sets.

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